CALCULATION OF AUTOOSCILLATIONS OF THE TYPE OF AZIMUTHAL WAVES OCCURRING AT LOSS OF STABILITY OF FLOW OF A VISCOUS FLUID BETWEEN CONCENTRIC CYLINDERS ROTATING IN OPPOSITE DIRECTIONS

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Starting with the Navier–Stokes equation we use the Lyapunov–Schmidt method to investigate the nature of the loss of stability of Couette flow between cylinders as the Reynolds number passes through its critical value. We consider the rotation of the cylinders in opposite directions with the ratio of the angular velocities such that the role of the most dangerous disturbances passes over from rotationally symmetric to nonrotationally symmetric disturbances. Branching nonstationary secondary flows (autooscillations) are found in the form of azimuthal waves; the longitudinal wave number α and the azimuthal wave number m are assumed given. The amplitude of autooscillations and the wave velocity are calculated for m = 1, and it is shown that depending on the value of α both weak excitation of stable and strong excitation of unstable autooscillations are possible and the wave number α for which the critical Reynolds number is a minimum corresponds to a stable wave regime in the supercritical region. The linear problem of the stability of the circular flow of a viscous fluid with respect to nonrotationally symmetric disturbances is discussed in [1-3]. Di Prima [1] solved the problem numerically by the Galerkin method when the gap is small and the cylinders rotate in the same direction. Di Prima's analysis is extended in [2] to cylinders rotating in opposite directions, and in [3] it is extended to gaps which are not small. The nonlinear stability problem is treated in [4], where for fixed $\alpha = 3$ and cylinders rotating in opposite directions the axisymmetric stationary secondary flow – the Taylor vortex – is calculated. The formation of azimuthal waves in the fluid between the cylinders was studied experimentally in detail by Coles [5].

1. Statement of the Problem

Suppose a viscous incompressible fluid of density ρ and kinematic viscosity ν fills the space between two coaxial cylinders of radii r_1 and r_2 ($r_1 < r_2$) which rotate with angular velocities Ω_1 and Ω_2, respectively. We take as units of length, time, and mass the quantities $r_2 - r_1$, $(r_2 - r_1)^2/\nu$, and $(r_2 - r_1)^3/\rho$, and introduce the Reynolds number $\mathrm{Re} = \Omega_1 r_1 (r_2 - r_1)/\nu$ and the dimensionless parameters $\mu = \Omega_2/\Omega_1$ and $\xi = r_1/(r_2 - r_1)$. The solution of the Gromek–Lamb form of the dimensionless equations of motion of the fluid

$$\mathbf{v}'_t + \boldsymbol{\omega}' \times \mathbf{v}' + \operatorname{grad} h' + \operatorname{rot} \boldsymbol{\omega}' = 0,\ \boldsymbol{\omega}' = \operatorname{rot} \mathbf{v}',\ \operatorname{div} \mathbf{v}' = 0 \tag{1.1}$$

in cylindrical coordinates r, θ, z is

$$\begin{gathered}\mathbf{v}' = \mathrm{Re}\,\mathbf{V},\ \boldsymbol{\omega}' = \mathrm{Re}\,\boldsymbol{\Omega},\ h' = h_0(r) = 0{,}5V_\theta^2 + \int V_\theta^2/r\,dr,\\ \mathbf{V} = (0, V_\theta, 0),\ \ \boldsymbol{\Omega} = (0, 0, \Omega_z),\ \ V_\theta = ar + b/r,\ \Omega_z = 2a,\\ a = (\mu(1+\xi)^2/\xi - \xi)/(1+2\xi),\ b = \xi(1-\mu)(1+\xi)^2/(1+2\xi),\end{gathered} \tag{1.2}$$

corresponding to laminar circular Couette flow. Since we are interested in flows which are periodic in time, branching from the solution (1.2), we set

$$\mathbf{v}' = \mathrm{Re}\,\mathbf{V} + \mathbf{v}(r, \tau, z),\ \boldsymbol{\omega}' = \mathrm{Re}\,\boldsymbol{\Omega} + \boldsymbol{\omega}(r, \tau, z),\ h' = h_0 + h(r, \tau, z)$$

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in (1.1), where $\tau = \theta - ct$ and c is the unknown velocity of azimuthal waves, and replace $\partial/\partial\theta$ by $\partial/\partial\tau$ and $\partial/\partial t$ by $-c\partial/\partial\tau$ to obtain the nonlinear eigenvalue problem

$$-c\mathbf{v}_\tau + \mathrm{Re}\,(\mathbf{\Omega}\times\mathbf{v} + \boldsymbol{\omega}\times\mathbf{V}) + \mathrm{grad}\,h + \mathrm{rot}\,\boldsymbol{\omega} = \mathbf{v}\times\boldsymbol{\omega}, \qquad (1.3)$$
$$\boldsymbol{\omega} = \mathrm{rot}\,\mathbf{v},\ \mathrm{div}\,\mathbf{v} = 0,$$

whose nontrivial solution must satisfy the nonslip condition $v = 0$ at $r = \xi$ and $r = 1 + \xi$ and have a zero velocity flux through the cross section of the cylindrical gap. We seek a solution having a period $2\pi/m$ in τ and $2\pi/\alpha$ in z, where m and α are the given azimuthal and longitudinal wave numbers, for which v_r, v_θ, h, and ω_z are even and v_z, ω_r, and ω_θ are odd functions of z. The remaining solutions can be obtained by shifting the origin along the axis of the cylinders.

2. The Lyapunov–Schmidt Method. Separation of Variables

We calculate autooscillating secondary flow by using the Lyapunov–Schmidt method [6, 7]. Setting

$$(\mathbf{v}, h, \boldsymbol{\omega}) = \sum_{n=1}^{\infty} \varepsilon^n (\mathbf{v}_n, h_n, \boldsymbol{\omega}_n),\ c = \mathrm{Re}_0 c_0/\xi + \sum_{n=1}^{\infty} \varepsilon^n c_n \qquad (2.1)$$

in (1.3), where $\epsilon = (\mathrm{Re} - \mathrm{Re}_0)^{1/2}$, Re_0 is the critical Reynolds number, and c_0 is the velocity of azimuthal waves relative to the rotating inner cylinder calculated from the linear theory, we obtain a set of linear problems of the form

$$\mathrm{Re}_0(-c_0/\xi\mathbf{v}_{n\tau} + \mathbf{\Omega}\times\mathbf{v}_n + \boldsymbol{\omega}_n\times\mathbf{V}) + \mathrm{grad}\,h_n + \mathrm{rot}\,\boldsymbol{\omega}_n = \mathbf{f}_n, \qquad (2.2)$$
$$\boldsymbol{\omega}_n = \mathrm{rot}\,\mathbf{v}_n,\ \mathrm{div}\,\mathbf{v}_n = 0,\ \mathbf{v}_n = 0\ (r = \xi,\ 1+\xi),$$

where the $\mathbf{f}_n$ are the known right-hand sides. In particular,

$$\mathbf{f}_1 = 0,\ \mathbf{f}_2 = \mathbf{v}_1\times\boldsymbol{\omega}_1 + c_1\mathbf{v}_{1\tau},\ \mathbf{f}_3 = c_1\mathbf{v}_{2\tau} + c_2\mathbf{v}_{1\tau} + \mathbf{v}_1\times\boldsymbol{\omega}_2 + \mathbf{v}_2\times\boldsymbol{\omega}_1 - (\mathbf{\Omega}\times\mathbf{v}_1 + \boldsymbol{\omega}_1\times\mathbf{V}).$$

For $n = 1$, a linear homogeneous problem is obtained for the calculation of the critical parameters Re_0, c_0, and the eigenfunction. We introduce the notation

$$e_{k,s} = \exp i(km\tau + s\alpha z),\ \mathbf{W}_{k,s} = (\mathbf{v}^{(k,s)}(r),\ h^{(k,s)}(r),\ \boldsymbol{\omega}^{(k,s)}(r))$$

and write the solution in the form

$$(\mathbf{v}_1, h_1, \boldsymbol{\omega}_1) = \beta(\mathbf{W}_{1,1}e_{1,1} + \overline{\mathbf{W}}_{1,1}\bar{e}_{1,1} + \mathbf{W}_{1,-1}e_{1,-1} + \overline{\mathbf{W}}_{1,-1}\bar{e}_{1,-1}),$$

where $\beta > 0$ is the amplitude of autooscillations [6] and a bar over a quantity denotes its complex conjugate, The parity with respect to z goes over into the parity with respect to s for the corresponding components of the vector $\mathbf{W}_{k,s}$. Separating the variables τ and z and introducing the notation $W_{1,1} = (\varphi, g, \gamma)$, we obtain directly from (2.2) for $n = 1$ the closed system of differential equations

$$D\varphi_r = -(\varphi_r + im\varphi_\theta)/r - i\alpha\varphi_z,\ D\varphi_\theta = (im\varphi_r - \varphi_\theta)/r + \gamma_z,$$
$$D\varphi_z = i\alpha\varphi_r - \gamma_\theta,\ Dg = \mathrm{Re}_0\,(imc_0/\xi\varphi_r + \Omega_z\varphi_\theta + V_\theta\gamma_z) - im/r\gamma_z + i\alpha\gamma_\theta,$$
$$D\gamma_\theta = \mathrm{Re}_0\,(imc_0/\xi\varphi_z - V_\theta\gamma_r) - i\alpha g + (im\gamma_r - \gamma_\theta)/r, \qquad (2.3)$$
$$D\gamma_z = \mathrm{Re}_0\,(\Omega_z\varphi_r - imc_0/\xi\varphi_\theta) + im/rg + i\alpha\gamma_r,$$
$$\gamma_r = im/r\varphi_z - i\alpha\varphi_\theta,\ D = d/dx,\ x = r - \xi,$$

for which we seek a nontrivial solution satisfying the boundary conditions $\varphi_r = \varphi_\theta = \varphi_z = 0$ at $x = 0$ and $x = 1$. Introducing the convenient normalization $\gamma_z = 0.25$ at $x = 0$ we note that for small ϵ the principal part of the tangential stress $p_{r\theta}$ at the inner cylinder varies according to the law $p_{r\theta} = \epsilon\beta \cos\alpha z \cos m \cdot (\theta - \mathrm{Re}_0 c_0/\xi t)$ with amplitude $\epsilon\beta$.

To construct the adjoint problem we multiply the first of Eqs. (2.2) for $n = 1$ scalarly by the solenoidal vector $\mathbf{\Psi}$ which vanishes at $r = \xi$ and $r = 1 + \xi$ and satisfies the same periodicity conditions. We then integrate by parts over the limits $\xi \le r \le 1+\xi$, $0 \le \tau \le 2\pi/m$, $0 \le z \le 2\pi/\alpha$ with the weight factor r and throw out products of $\mathbf{\Psi}$ and derivatives with v_1, h_1, and ω_1. As a result we obtain the adjoint problem

$$\mathrm{Re}_0\,(c_0/\xi\mathbf{\Psi}\tau + \mathbf{\Psi}\times\mathbf{\Omega}) + \mathrm{grad}\,Q + \mathrm{rot}\,\mathbf{\Lambda} = 0,\ \mathrm{div}\,\mathbf{\Psi} = 0,$$
$$\mathrm{rot}\,\mathbf{\Psi} + \mathrm{Re}_0\,\mathbf{V}\times\mathbf{\Psi} = \mathbf{\Lambda},\ \mathbf{\Psi} = 0\ (r = \xi,\ 1+\xi),$$

which after separating the variables τ and z

$$(\mathbf{\Psi}, Q, \mathbf{\Lambda}) = (\boldsymbol{\psi}, q, \boldsymbol{\lambda})e_{1,1}$$

reduces to the ordinary differential equations

$$D\psi_r = -(\psi_r + im\psi_\theta)/r - i\alpha\psi_z,\ D\psi_\theta = (im\psi_r - \psi_\theta)/r + \mathrm{Re}_0 V_\theta\psi_r + \lambda_z,\ D\psi_z =$$
$$= i\alpha\psi_r - \lambda_\theta,\ Dq = -im\,\mathrm{Re}_0\, c_0/\xi\psi_r - \mathrm{Re}_0\,\Omega_z\psi_\theta - im/r\lambda_z + i\alpha\lambda_\theta,$$
$$D\lambda_\theta = -im\,\mathrm{Re}_0\, c_0/\xi\psi_z - i\alpha q + (im\lambda_r - \lambda_\theta)/r, \tag{2.4}$$
$$D\lambda_z = im\,\mathrm{Re}_0\, c_0/\xi\psi_\theta - \mathrm{Re}_0\,\Omega_z\psi_r + im/r\,q + i\alpha\lambda_r,$$
$$\lambda_r = (im/r + \mathrm{Re}_0\, V_\theta)\psi_z - i\alpha\psi_\theta$$

with the boundary conditions $\psi_r = \psi_\theta = \psi_z = 0$ at $x = 0$ and $x = 1$ and the auxiliary normalization condition $\lambda_z = 0.25$ at $x = 0$. The condition that the inhomogeneous problem (2.2) be solvable,

$$\int_0^{2\pi/\alpha}\int_0^{2\pi/m}\int_\xi^{1+\xi} (\mathbf{f}_n, \psi)\,\bar{e}_{1,1} r\,dr\,d\tau\,dz = 0\quad (n = 2, 3, 4, \ldots), \tag{2.5}$$

shows that $c_1 = 0$ for $n = 2$ if the quantity

$$I_1 = \int_\xi^{1+\xi} (\varphi, \psi)\, r\,dr$$

is different from zero. Assuming that this condition is satisfied we seek the solution of problem (2.2) for $n = 2$ to fit the right-hand side $\mathbf{f}_2$ in the form

$$(\mathbf{v}_2, h, \omega_2) = \beta^2(\mathbf{W}_{0,0}e_{0,0} + \mathbf{W}_{2,2}e_{2,2} + \overline{\mathbf{W}}_{2,2}\bar{e}_{2,2} + \mathbf{W}_{2,-2}e_{2,-2} +$$
$$+ \overline{\mathbf{W}}_{2,-2}\bar{e}_{2,-2} + \mathbf{W}_{2,0}e_{2,0} + \overline{\mathbf{W}}_{2,0}\bar{e}_{2,0} + \mathbf{W}_{0,2}e_{0,2} + \overline{\mathbf{W}}_{0,2}\bar{e}_{0,2}), \tag{2.6}$$

where it is assumed that the components of the vectors $\mathbf{W}_{k,s}$ have the same parity with respect to s as they had with respect to z before the separation of variables. We note that a solution of the linear homogeneous equation should be added to the right-hand side of Eq. (2.6) with a constant coefficient β', but as shown in [7] for the general case of branching $\beta' = 0$. The parity with respect to the second subscript enables us to limit ourselves to finding only the four vector functions $\mathbf{W}_{0,0}$, $\mathbf{W}_{2,2}$, $\mathbf{W}_{2,0}$, and $\mathbf{W}_{0,2}$. The components of the first satisfy the equations

$$Dv_\theta^{(0,0)} = \omega_z^{(0,0)} - v_\theta^{(0,0)}/r,\ D\omega_z^{(0,0)} = -A_\theta^{(0,0)},\ v_\theta^{(0,0)} = 0\ (x = 0,1),$$
$$Dh^{(0,0)} = \mathrm{Re}_0\left(\Omega_z v_\theta^{(0,0)} + V_\theta\omega_z^{(0,0)}\right) + A_r^{(0,0)},\ v_r^{(0,0)} = v_z^{(0,0)} = \omega_r^{(0,0)} = \omega_\theta^{(0,0)} \equiv 0, \tag{2.7}$$
$$\left(A_r^{(0,0)}, A_\theta^{(0,0)}, 0\right) = \mathbf{v}^{(1,1)} \times \bar{\omega}^{(1,1)} + \bar{\mathbf{v}}^{(1,1)} \times \omega^{(1,1)} + \mathbf{v}^{(1,-1)}\ \bar{\omega}^{(1,-1)} + \bar{\mathbf{v}}^{(1,-1)} \times \omega^{(1,-1)}.$$

The arbitrary constant in the determination of the total pressure is conveniently fixed by setting $h^{0,0} = 0$ at $x = 1$.

For the components of the vector $\mathbf{W}_{2,2}$ we obtain

$$Dv_r^{(2,2)} = -\left(v_r^{(2,2)} + 2\,im v_\theta^{(2,2)}\right)/r - 2i\alpha v_z^{(2,2)},\quad Dv_\theta^{(2,2)} = \left(2im v_r^{(2,2)} - v_\theta^{(2,2)}\right)/r + \omega_z^{(2,2)},$$
$$Dv_z^{(2,2)} = 2i\alpha v_r^{(2,2)} - \omega_\theta^{(2,2)},\ Dh^{(2,2)} = 2im\,\mathrm{Re}_0 c_0/\xi v_r^{(2,2)} + \mathrm{Re}_0\left(\Omega_z v_\theta^{(2,2)} + V_\theta\omega_z^{(2,2)}\right) - 2im/r\omega_z^{(2,2)} + 2i\alpha\omega_\theta^{(2,2)} + A_r^{(2,2)},$$
$$D\omega_\theta^{(2,2)} = 2im\,\mathrm{Re}_0 c_0/\xi v_z^{(2,2)} - \mathrm{Re}_0 V_0\omega_r^{(2,2)} - 2i\alpha h^{(2,2)} + \left(2im\omega_r^{(2,2)} - \omega_\theta^{(2,2)}\right)/r + A_z^{(2,2)}, \tag{2.8}$$
$$D\omega_z^{(2,2)} = -2im\,\mathrm{Re}_0\, c_0/\xi v_\theta^{(2,2)} + \mathrm{Re}_0\,\Omega_z v_r^{(2,2)} + 2im/rh^{(2,2)} + 2i\alpha\omega_r^{(2,2)} - A_\theta^{(2,2)},$$
$$\omega_r^{(2,2)} = 2im/rv_z^{(2,2)} - 2i\alpha v_\theta^{(2,2)},\ A^{(2,2)} = \mathbf{v}^{(1,1)} \times \omega^{(1,1)},\ v_r^{(2,2)} = v_\theta^{(2,2)} = v_z^{(2,2)} = 0\ (x = 0,1).$$

We find the components of the vector $\mathbf{W}_{2,2}$ by solving the problem

$$Dv_r^{(2,0)} = -\left(v_r^{(2,0)} + 2im v_\theta^{(2,0)}\right)/r,\ Dv_\theta^{(2,0)} = \left(2im v_r^{(2,0)} - v_\theta^{(2,0)}\right)/r + \omega_z^{(2,0)}, \tag{2.9}$$
$$Dh^{(2,0)} = 2im\,\mathrm{Re}_0\, c_0/\xi v_r^{(2,0)} + \mathrm{Re}_0\left(\Omega_z v_\theta^{(2,0)} + V_\theta\omega_z^{(2,0)}\right) - 2im/r\omega_z^{(2,0)} + A_r^{(2,0)},$$
$$D\omega_z^{(2,0)} = -2im\,\mathrm{Re}_0\, c_0/\xi v_\theta^{(2,0)} + \mathrm{Re}_0\Omega_z v_r^{(2,0)} + 2im/rh^{(2,0)} - A_\theta^{(2,0)},$$
$$\omega_r^{(2,0)} = \omega_\theta^{(2,0)} = v_z^{(2,0)} \equiv 0,\ \left(A_r^{(2,0)}, A_\theta^{(2,0)}, 0\right) = \mathbf{v}^{(1,1)} \times \omega^{(1,-1)} + \mathbf{v}^{(1,-1)} \times \omega^{(1,1)}$$

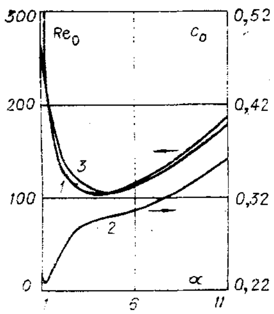

Fig. 1

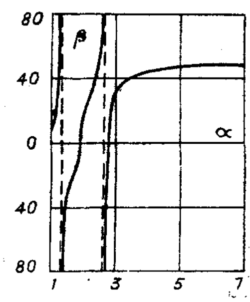

Fig. 2

TABLE 1

α	Re_0	c_0	β	c_2	$Real \sigma_2$	$Im \sigma_2$
3,930	103,52	0,29672	43.31	0,303	0,4165	—0,3229
1,075	367,09	0,23007	7,846	0,236	0,0590	—0,2944
1,275	231,72	0,23042	40,53	—0,200	0,1028	—0,2816
1,325	215,89	0,23231	94,01	—2,15	0,1131	—0,2824
1,350	209,14	0,23333	115,2*	—3,88	0,1180	—0,2830
1,600	165,58	0,24428	24,87*	—0,472	0,1614	—0,2917
1,850	142,93	0,25472	11,45*	—0,492	0,1973	—0,3004
1,950	136,65	0,25857	13,66	0,103	0,2103	—0,3034
2,300	121,58	0,27040	33,85	—0,077	0,2520	—0,3115
2,600	113,88	0,27852	65,74	—1,90	0,2847	—0,3159
2,700	111,99	0,28084	81,83*	—5,65	0,2952	—0,3170
2,800	110,38	0,28296	20,09*	—1,47	0,3055	—0.3179
3,000	107,82	0,28668	29,89	0,715	0,3258	—0,3192
5,000	106,95	0,30123	46,38	0,204	0,5139	—0,3320
6,600	120,66	0,30998	47,98	0,177	0,6389	—0,3584

with the boundary conditions $v_r^{(2,0)} = v_\theta^{(2,0)} = 0$ at $x = 0$ and $x = 1$. For the components of the vector $W_{0,2}$ we obtain the equations

$$Dv_r^{(0,2)} = -v_r^{(0,2)}/r - 2i\alpha v_z^{(0,2)},\ Dv_\theta^{(0,2)} = -v_\theta^{(0,2)}/r + \omega_z^{(0,2)}, \tag{2.10}$$

$$Dv_z^{(0,2)} = 2i\alpha v_r^{(0,2)} - \omega_\theta^{(0,2)},\ Dh^{(0,2)} = \mathrm{Re}_0\left(\Omega_z v_\theta^{(0,2)} + V_\theta \omega_z^{(0,2)}\right) + 2i\alpha\omega_\theta^{(0,2)} + A_r^{(0,2)},$$

$$D\omega_\theta^{(0,2)} = -\mathrm{Re}_0 V_\theta \omega_r^{(0,2)} - 2i\alpha h^{(0,2)} - \omega_\theta^{(0,2)}/r + A_z^{(0,2)},$$

$$D\omega_z^{(0,2)} = \mathrm{Re}_0 \Omega_z v_r^{(0,2)} + 2i\alpha\omega_r^{(0,2)} - A_\theta^{(0,2)},\ \omega_r^{(0,2)} = -2i\alpha v_\theta^{(0,2)},$$

$$\mathbf{A}^{(0,2)} = \mathbf{v}^{(1,1)} \times \bar{\omega}^{(1,-1)} + \bar{\mathbf{v}}^{(1,-1)} \times \omega^{(1,1)},\ v_r^{(0,2)} = v_\theta^{(0,2)} = v_z^{(0,2)} = 0\ (x=0,1),$$

where it turns out that the quantities $v_r^{(0,2)}$, $v_\theta^{(0,2)}$, $h^{(0,2)}$, and $\omega_z^{(0,2)}$ are real and $v_z^{(0,2)}$, $\omega_r^{(0,2)}$, and $\omega_\theta^{(0,2)}$ are purely imaginary. Setting $n = 3$ in the solvability condition (2.5) we obtain an equation for the real constants β and c_2:

$$imc_2 I_1 + \beta^2 I_2 = I_3,$$

$$I_2 = \int_{\xi}^{1+\xi} (\chi, \psi)\, r dr,\ I_3 = \int_{\xi}^{1+\xi} (\Omega \times \varphi + \gamma \times \mathbf{V}, \psi)\, r dr,$$

$$\chi = \mathbf{v}^{(1,1)} \times \omega^{(0,0)} + \bar{\mathbf{v}}^{(1,1)} \times \omega^{(2,2)} + \mathbf{v}^{(1,-1)} \times \omega^{(0,2)} + \bar{\mathbf{v}}^{(1,-1)} \times \omega^{(2,0)} +$$
$$+ \mathbf{v}^{(0,0)} \times \omega^{(1,1)} + \mathbf{v}^{(2,2)} \times \bar{\omega}^{(1,1)} + \mathbf{v}^{(0,2)} \times \omega^{(1,-1)} + \mathbf{v}^{(2,0)} \times \bar{\omega}^{(1,-1)},$$

from which we find

$$\beta^2 = d_1 \equiv \mathrm{Real}\,(I_3\bar{I}_1)/\mathrm{Real}\,(I_1\bar{I}_2),\ c_2 = d_2 \equiv \mathrm{Im}\,(I_3\bar{I}_2)/m\,\mathrm{Real}\,(I_1\bar{I}_2).$$

If the constant d_1 is greater than zero, autooscillations branch off for $Re > Re_0$ (supercritical autooscillations). For $d_1 < 0$ the equation for β^2 becomes inconsistent. In this case it is necessary to consider subcritical values of the Reynolds number, setting $Re = Re_0 - \epsilon^2$ in the derivation of the chain of equations (2.2). As a consequence $I_3 \to -I_3$. This gives the correct value of the square of the amplitude corresponding now to subcritical autooscillations. Both variants of the branching can be described simultaneously if we evaluate the constants d_1 and d_2 starting from the assumption $Re = Re_0 + \epsilon^2$ and then setting

$$\beta = |d_1|^{1/2},\ c_2 = d_2\,\mathrm{sgn}\,d_1,\ \mathrm{Re} = \mathrm{Re}_0 + \varepsilon^2\,\mathrm{sgn}\,d_1.$$

To investigate the stability of circular Couette flow and branching of autooscillations we use the method of linearization. The validity of linearization in problems of the stability of stationary and periodic motions of a fluid is shown in [8-10]. This leads to the spectral problem

$$\sigma \mathbf{u} - \mathrm{Re}_0\, c_0/\xi \mathbf{u}_\tau + \mathrm{Re}\,(\Omega \times \mathbf{u} + \mathrm{rot}\,\mathbf{u} \times \mathbf{V}) + \mathrm{grad}\,p + \mathrm{rot}\,\mathrm{rot}\,\mathbf{u} = 0,$$
$$\mathrm{div}\,\mathbf{u} = 0,\ \mathbf{u} = 0\ (r = \xi,\ 1 + \xi)$$

for Couette flow, and to the problem

$$\sigma' \mathbf{u}' - c\mathbf{u}'_\tau + \mathrm{Re}\,(\Omega \times \mathbf{u}' + \mathrm{rot}\,\mathbf{u}' \times \mathbf{V}) + \omega \times \mathbf{u}' + \mathrm{rot}\,\mathbf{u}' \times \mathbf{v} + \mathrm{grad}\,p' +$$
$$+ \mathrm{rot}\,\mathrm{rot}\,\mathbf{u}' = 0,\ \mathrm{div}\,\mathbf{u}' = 0, \mathbf{u}' = 0\ (r = \xi, 1 + \xi)$$

for the autooscillation regime. Here c, v, and ω are determined according to (2.1). Applying the perturbation

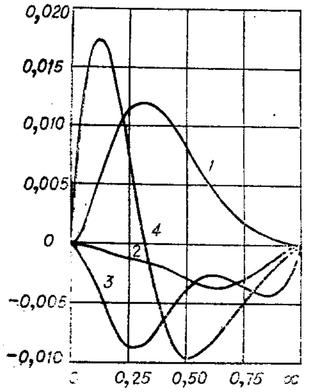

Fig. 3

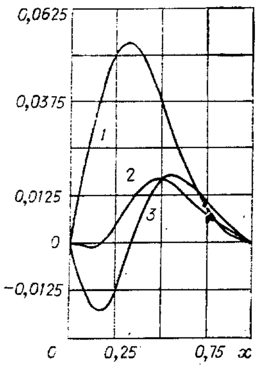

Fig. 4

method [7] for Reynolds numbers close to the critical value we find the eigenvalues of σ and σ':

$$\sigma = \sigma_2(\mathrm{Re} - \mathrm{Re}_0) + 0(|\mathrm{Re} - \mathrm{Re}_0|^2), \quad \sigma' = \sigma_2'\varepsilon^2 + 0(\varepsilon^4), \qquad (2.11)$$
$$\sigma_2 = -I_3/I_1, \quad \mathrm{Real}\,\sigma_2' = -\mathrm{Real}\,\sigma_2 \,\mathrm{sgn}\, d_1, \quad \mathrm{Re} = \mathrm{Re}_0 + \varepsilon^2 \,\mathrm{sgn}\, d_1.$$

The stability of the corresponding flow can be judged from the real parts of these.

Suppose the following conditions are satisfied:

$$\mathrm{Real}\,\sigma_2 > 0, \quad \mathrm{Real}\,(I_3\bar{I}_1) \neq 0, \quad \mathrm{Real}\,(I_1\bar{I}_2) \neq 0, \qquad (2.12)$$

then on the basis of Theorems 2.2 and 3.1 of [6, 7] it can be stated that for $\mathrm{Re} > \mathrm{Re}_0$ Couette flow loses stability and at the point $\mathrm{Re} = \mathrm{Re}_0$ the autooscillating regime having the form of azimuthal waves branches off from it. The branching out of a cycle occurs in the supercritical region $\mathrm{Re} > \mathrm{Re}_0$ if $d_1 > 0$ or in the subcritical region $\mathrm{Re} < \mathrm{Re}_0$ if $d_1 < 0$. It is clear from (2.11) that supercritical autooscillations are stable and subcritical oscillations unstable.

3. Results of Calculations

The secondary wave regime was calculated on an ODRA-1204 computer for mode $m = 1$, fixed values of the parameters $\xi = 1$, $\mu = -0.45$, and various longitudinal wave numbers $1 \le \alpha \le 7$ at 58 points altogether. The critical values of the parameters Re_0 and c_0 for $m = 1$ and $m = 0$ were accurately calculated in advance by direct numerical integration of (2.3). The law of motion for α was used together with Newton's method for solving a system of two transcendental equations by approximating partial derivatives by finite differences. The results of the calculations are shown in Fig. 1, where the numbers 1 and 2 denote the curves for $\mathrm{Re}_0(\alpha)$ and $c_0(\alpha)$ for $m = 1$; the number 3 marks the neutral curve for $\mathrm{Re}_0(\alpha)$ corresponding to a rotationally symmetric disturbance with $m = 0$, $c_0 \equiv 0$. It is clear from Fig. 1 that for the chosen ratio of the angular velocities of the cylinders nonrotationally symmetric disturbances are more dangerous than those which are rotationally symmetric; on curve 1 ($m = 1$) the minimum is reached at the point $\alpha = 3.93$, $\mathrm{Re}_0 = 103.52$, and on curve 3 ($m = 0$) at the point $\alpha = 4.51$, $\mathrm{Re}_0 = 105.95$. After finding the eigenvalues of Re_0 and c_0 the boundary-value problems (2.3), (2.4), (2.7)-(2.10) were solved and the quantities β, c_2, σ_2, and Real σ_2' were calculated.

Conditions (2.12) were satisfied at the same time. In view of the small value of the Reynolds numbers all the boundary-value problems were solved by taking linear combinations of particular solutions obtained by integrating several Cauchy problems from the point $x = 1$ to the point $x = 0$ by the standard fourth-order Runge–Kutta method with automatic step selection. The boundary condition on the inner cylinder requiring the vanishing of the azimuthal component of the velocity for the homogeneous problems (2.3), (2.4) was replaced by the normalization condition. The accuracy of the previously found eigenvalues determined how well the discarded boundary condition was satisfied. To make the programming more convenient the evaluation of the integrals I_1, I_2, and I_3 was reduced to a Cauchy problem with a zero initial condition on the outer cylinder. The method described for calculating autooscillations required the simultaneous integration of 61 first-order differential equations in the final stage.

The dependence of the constant β on α is shown graphically in Fig. 2, where for clarity the positive values of β corresponding to subcritical autooscillations are plotted below the α axis. The graph has two asymptotes of the form $\alpha = \alpha_a$. As these points are approached the constant β increases without bound, behaving as $\beta \sim \mathrm{const}\,|\alpha - \alpha_a|^{-1/2}$. At the two points $\alpha = \alpha_k$ ($k = 1,2$) the amplitude vanishes: $\beta \sim \mathrm{const}_k\,|\alpha =$

$\alpha_k|^{1/2}(\alpha \to \alpha_k, k=1,2)$. At these four exceptional points the conditions (2.12) are not satisfied and expansions (2.1) no longer hold. Some numerical results are shown in Table 1, where an asterisk denotes a number corresponding to the subcritical case. The minimum is the most interesting point of the neutral curve, since the first loss of stability occurs in the passage through this point. There is a branching off of a stable periodic regime, a weak excitation of autooscillations of the type of azimuthal waves with parameters listed in the first row of Table 1. The graphs of certain components of the solution corresponding to the numbers in this row are shown in Fig. 3: 1) Real φ_r; 2) Im φ_r; 3) Real φ_z; 4) Im φ_z, and Fig. 4: 1) Real φ_θ; 2) Im φ_θ; 3) 100 $v_\theta^{(0,0)}$.

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